A Generalization of Order Theoretic Properties of the Line and Topological Implications

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Abstract

We study properties of nests in order to examine order-theoretic properties of linearly ordered sets, in particular LOTS, and we investigate their topological implications. We expand these properties to sets that are not necessarily linearly ordered.

Keywords: orderability problem, linearly ordered set, LOTS, interval topology, nest, T_0-separation, T_1-separation

1. Introduction.

In S. Purisch's account of results on orderability and suborderability (see [1]), one can read the formulation and development of several orderability problems, starting from the beginning of the 20th century and reaching our days. By an orderability problem, in topology, we mean the following. Let $(X, \mathcal{T})$ be a topological space and let X be equipped with an order relation $<$. Under what conditions will $\mathcal{T}_<$, i.e. the topology induced by the order $<$, be equal to $\mathcal{T}$?

The first general solution to the characterization of LOTS (linearly ordered topological spaces) was given by J. van Dalen and E. Wattel, in 1973 (see [2]). The authors considered a topological space X, whose topology is generated by a subbase $\mathcal{L} \cup \mathcal{R}$ of two nests $\mathcal{L}$ and $\mathcal{R}$ on X, whose union is T_1-separating [1]. By considering the ordering which is generated by the nest $\mathcal{L}$ on X, namely $\triangleleft_\mathcal{L}$, the authors introduced conditions such that $\mathcal{T}_{\triangleleft_\mathcal{L}}$ to be equal to $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$. In this case, they said the space to be LOTS, while in the

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[1] For explicit definitions see the preliminaries section 2.

case where $\mathcal{T}_{\triangleleft_{\mathcal{L}}} \subset \mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$, the space was said to be GO (generalized ordered). Both in the case of GO-spaces and of LOTS the authors demanded $\mathcal{L} \cup \mathcal{R}$ to form a T_1-separating subbase for the topology on X (see Definition 2.3). The necessary and sufficient condition for both nests $\mathcal{L}$, $\mathcal{R}$ to be interlocking (see Definition 2.7) was added in order for the space to be LOTS. The authors of [3] re-investigated the property of interlocking, leading to a result (see Theorem 2.8) that we will use in this paper in order to re-examine this orderability theorem from another perpective (see theorems 5.1 and 5.2).

From the observation that a nest is T_0-separating, if and only if $\triangleleft_{\mathcal{L}}$ is a linear order, together with Theorem 2.6, one can deduce that J. van Dalen and E. Wattel restricted their study to lines, exclusively. Indeed, a T_1-separating subbase which consists of the union of two nests describes in a natural way the topology of a line X. In this paper we are inspired by the mentioned characterization of J. van Dalen and E. Wattel, but we use different conditions in order to extend order-theoretic properties of the line. We define the interval topology on a set X, using a nest $\mathcal{L}$ and its generated reflexive order $\trianglelefteq_{\mathcal{L}}$, and we investigate under what conditions this topology coincides with the topology generated by $\mathcal{L}\cup\mathcal{R}$. This time $\mathcal{L}\cup\mathcal{R}$ need not be T_1-separating, although $\trianglelefteq_{\mathcal{L}}$ will be still equal to $\trianglerighteq_{\mathcal{R}}$, a frame which will provide us with a set of dual conditions between the two nests. We observe that the interval topology, when it is defined via the generated order of a nest $\mathcal{L}$, introduces a lot of similarities with the topology that is build by the union of two "dual" [2] nests $\mathcal{L}$ and $\mathcal{R}$. While J. van Dalen and E. Wattel examined the comparison between topologies $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$ and $\mathcal{T}_{\triangleleft_{\mathcal{L}}}$, in order to conclude whether the space is LOTS or a subspace of a LOTS, we will examine the properties which give a comparison between $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$ and $\mathcal{T}^{\mathcal{L}}_{in}$, the interval topology induced by $\mathcal{L}$, in order to see whether the structure of a space is linear or not and, in particular, how close -or not- are its order-theoretic properties with the natural properties that are carried by the real line.

2. Preliminaries.

Here we introduce some basic notions. For a detailed discussion on upper sets, lower sets and the interval topology see [4]. For a detailed discussion on nests and orderings see [2] and the more recent developments [5] and [3].

Definition 2.1. Let $(X, <)$ be a partially ordered set. We define $\uparrow A \subset X$,

[2]see Definition 4.13

to be the set:

$$\uparrow A = \{x : x \in X \text{ and there exists } y \in A, \text{ such that } y < x\}.$$

We also define $\downarrow A \subset X$, to be the set:

$$\downarrow A = \{x : x \in X \text{ and there exists } y \in A, \text{ such that } x < y\}.$$

More specifically, if $A = \{y\}$, then:

$$\uparrow A = \{x : x \in X \text{ and } y < x\}$$

and

$$\downarrow A = \{x : x \in X \text{ and } x < y\}$$

From now on we will use the conventions $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$.

We remind that the *upper topology* $\mathcal{T}_U$, on an ordered set $(X, <)$, is generated by the subbase $\mathcal{S} = \{X - \downarrow x : x \in X\}$ and the *lower topology* $\mathcal{T}_l$ is generated by the subbase $\mathcal{S} = \{X - \uparrow x : x \in X\}$. The *interval topology* $\mathcal{T}_{in}$ is defined as $\mathcal{T}_{in} = \mathcal{T}_U \vee \mathcal{T}_l$.

Definition 2.2. Let X be a set. A collection $\mathcal{L}$, of subsets of X, T_0-*separates* X, if and only if for all $x, y \in X$, such that $x \neq y$, there exist $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$ or $y \in L$ and $x \notin L$.

Definition 2.3. Let X be a set. A collection $\mathcal{L}$, of subsets of X, T_1-*separates* X, if and only if for all $x, y \in X$, such that $x \neq y$, there exist $L, L' \in \mathcal{L}$, such that $x \in L$ and $y \notin L$ and also $y \in L'$ and $x \notin L'$.

One can easily see the link between definitions 2.2 and 2.3 and the separation axioms of topology: a topological space $(X, \mathcal{T})$ is T_0 (resp. T_1), if and only if every subbase $\mathcal{S}$, for $\mathcal{T}$, T_0-separates (resp. T_1-separates) X.

Definition 2.4. Let X be a set and let $\mathcal{L} \subset X$ be a family of subsets of X. $\mathcal{L}$ is a *nest* on X, if for every $M, N \in \mathcal{L}$, either $M \subset N$ or $N \subset M$.

Definition 2.5. Let X be a set and let $\mathcal{L}$ be a nest on X. We define an order relation on X via the nest $\mathcal{L}$, as follows:

$$x \lhd_{\mathcal{L}} y \Leftrightarrow \exists L \in \mathcal{L}, \text{ such that } x \in L \text{ and } y \notin L$$

The order of Definition 2.5 was first introduced in [2] and was further examined in [3].

Obviously, if $x, y \in X$, then $x \trianglelefteq_{\mathcal{L}} y$, if and only if either $x = y$ or there exists $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$.

From now on, whenever we write $x \triangleleft_{\mathcal{L}} y$, we will assume that $x \neq y$.

A useful theorem in this paper will be the following (see [3]).

Theorem 2.6. *Let X be a set. Suppose $\mathcal{L}$ and $\mathcal{R}$ are two nests on X. $\mathcal{L} \cup \mathcal{R}$ is T_1-separating, if and only if $\mathcal{L}$ and $\mathcal{R}$ are both T_0-separating and $\triangleleft_{\mathcal{L}} = \triangleright_{\mathcal{R}}$.*

The following notion of interlocking will be useful for theorems 5.1 and 5.2.

Definition 2.7 (van Dalen & Wattel)**.** Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$. We say that $\mathcal{L}$ is interlocking if and only if, for each $L \in \mathcal{L}$, $L = \bigcap\{N \in \mathcal{L} : L \subsetneq N\}$ implies $L = \bigcup\{N \in \mathcal{L} : N \subsetneq L\}$.

Theorem 2.8 (See [3])**.** *Let X be a set and let $\mathcal{L}$ be a T_0-separating nest on X. The following are equivalent:*

1. *$\mathcal{L}$ is interlocking;*
2. *for each $L \in \mathcal{L}$, if L has a $\triangleleft_{\mathcal{L}}$-maximal element, then $X - L$ has a $\triangleleft_{\mathcal{L}}$-minimal element;*
3. *for all $L \in \mathcal{L}$, either L has no $\triangleleft_{\mathcal{L}}$-maximal element or $X - L$ has a $\triangleleft_{\mathcal{L}}$-minimal element.*

3. A Close Up to the Interval Topology via $\trianglelefteq_{\mathcal{L}}$, when $\mathcal{L}$ is T_0-separating.

Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests on X, such that $\mathcal{L} \cup \mathcal{R}$ T_1-separates X. According to Theorem 2.6 each of $\mathcal{L}$ and $\mathcal{R}$ are T_0-separating, so $\triangleleft_{\mathcal{L}} = \triangleright_{\mathcal{R}}$ is a linear order.

Construction. We consider the lower topology on X, with respect to $\trianglelefteq_{\mathcal{L}}$. We denote this topology by $\mathcal{T}_l^{\trianglelefteq_{\mathcal{L}}}$. Then, for each $y \in X$, $\uparrow y = \{x \in X : y \trianglelefteq_{\mathcal{L}} x\}$. So, $X - \uparrow y = \{x \in X : x \triangleleft_{\mathcal{L}} y\}$. This happens, because $\mathcal{L}$ is a T_0-separating nest. Thus, a subbase for the lower topology on X, which is generated by $\trianglelefteq_{\mathcal{L}}$, will be of the form:

$$\mathcal{S}(\mathcal{T}_l^{\trianglelefteq_{\mathcal{L}}}) = \{X - \uparrow y : y \in X\}.$$

We now consider the upper topology on X, with respect to $\trianglelefteq_{\mathcal{L}}$. We denote this topology by $\mathcal{T}_U^{\trianglelefteq_{\mathcal{L}}}$. Then, for each $y \in X$, $\downarrow y = \{x \in X : x \trianglelefteq_{\mathcal{L}} y\}$. So, $X - \downarrow y = \{x \in X : y \triangleleft_{\mathcal{L}} x\}$. Thus, a subbase for the upper topology on X, that is generated by $\trianglelefteq_{\mathcal{L}}$, is of the form:

$$\mathcal{S}(\mathcal{T}_U^{\trianglelefteq_{\mathcal{L}}}) = \{X - \downarrow y : y \in X\}$$

We construct the interval topology which is generated by $\mathcal{L}$, denoted by $\mathcal{T}_{in}^{\mathcal{L}}$, as follows:

$$\mathcal{T}_{in}^{\mathcal{L}} = \mathcal{T}_U^{\trianglelefteq_{\mathcal{L}}} \vee \mathcal{T}_l^{\trianglelefteq_{\mathcal{L}}}.$$

A subbase for this topology will be:

$$\mathcal{S}_{in} = \mathcal{S}(\mathcal{T}_U^{\trianglelefteq_{\mathcal{L}}}) \cup \mathcal{S}(\mathcal{T}_l^{\trianglelefteq_{\mathcal{L}}}).$$

Remark 3.1. (We remind that $\mathcal{L}$, throughout this section, T_0-separates X.)

1. In our construction of the interval topology we used a reflexive order $\trianglelefteq_{\mathcal{L}}$, rather than a non-reflexive one $\triangleleft_{\mathcal{L}}$. This is because the non-reflexive $\triangleleft_{\mathcal{L}}$ will generate an interval topology equal to the discrete topology on X (a trivial case to study). Indeed, $\downarrow a = \{x \in X : x \triangleleft_{\mathcal{L}} a\}$ and so $X - \downarrow a = \{x \in X : a \trianglelefteq_{\mathcal{L}} x\} = (-\infty, a]$. In a similar fashion, $x - \uparrow a = [a, \infty)$ and so $(-\infty, a] \cap [a, \infty) = \{a\}$.
2. The sets in $\mathcal{T}_U^{\trianglelefteq_{\mathcal{L}}}$ form a nest and the sets in $\mathcal{T}_l^{\trianglelefteq_{\mathcal{L}}}$ also form a nest. It will be particularly useful to remember this, whenever we compare $\mathcal{T}_{in}^{\mathcal{L}}$ with $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$, in the corresponding section. It will be also useful to bare in mind that in the set of real numbers, equipped with its usual topology, $\mathcal{T}_{in}^{\mathcal{L}} = \mathcal{T}_{\mathcal{L}\cup\mathcal{R}} = \mathcal{T}_{\triangleleft_{\mathcal{L}}}$, where $\mathcal{L} = \{(-\infty, a) : a \in \mathbb{R}\}$ and $\triangleleft_{\mathcal{L}} = \triangleright_{\mathcal{R}}$.

4. A Generalization of Order Theoretic Properties of the Line.

Consider the set of real numbers $\mathbb{R}$, equipped with its usual topology. Let $\mathcal{L} = \{(-\infty, a) : a \in \mathbb{R}\}$. We remark that for each $(-\infty, a) \in \mathcal{L}$, $\sup L = a \notin L$. We also remark that for each $k \in \mathbb{R}$, there exists $L = (-\infty, k) \in \mathcal{L}$, such that $\sup L = k$. We will now generalize this remark to arbitrary sets. In particular, we will use the following three conditions, namely (C1), (C2), (C3), in order to investigate the relationship between the topologies $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$ and $\mathcal{T}_{in}^{\mathcal{L}}$; this relationship will be a measure of linearity, that is, it will show how close -or not- is a space from a LOTS, regarding its structure. From now on, sup will be used for abbreviating the term *supremum* and inf will abbreviate the term *infimum*.

Let $\mathcal{L}$ be a nest on a set X. We introduce the following three conditions:

(C1) For each $L \in \mathcal{L}$, there exists $\sup L$ with respect $\trianglelefteq_{\mathcal{L}}$.

(C2) For each $L \in \mathcal{L}$, there exists $\sup L$ with respect to $\trianglelefteq_{\mathcal{L}}$, such that $\sup L \in X - L$.

(C3) For each x, there exists $L \in \mathcal{L}$, such that there exists $\sup L = x \in X - L$ and also property (C2) holds.

We deduce the following relations between (C1), (C2) and (C3).

Proposition 4.1.

1. *(C3) implies (C2).*
2. *(C2) implies (C1).*
3. *(C1) does not always imply (C2).*
4. *(C2) does not always imply (C3).*
5. *(C3) implies that* $\mathcal{L}$ *is* T_0*-separating.*
6. $\mathcal{L}$ T_0*-separating implies neither (C1) nor (C2) nor (C3).*
7. *Neither (C1) nor (C2) imply that* $\mathcal{L}$ *is* T_0*-separating.*

Proof. The statement that (C3) implies (C2) follows immediately from the definition of (C3). Similarly, (C2) implies (C1) by the definition of (C2). Example 4.2 shows that (C1) does not always imply (C2) or T_0-separation. Example 4.3 shows that (C2) does not always imply (C3) or T_0-separation. Proposition 4.8 shows that (C3) implies T_0-separation. Examples 4.7, 4.6 and 4.5 show that the T_0-separation of $\mathcal{L}$ does not necessarily imply property (C1) or (C2) or (C3). $\square$

Example 4.2. Let $X = (0, 1)$ and consider the nest $\mathcal{L} = \{(0, a] : a \in \mathbb{R}, \frac{1}{2} \leq a < 1\}$, on X. We remark that condition (C1) is satisfied, but (C2) is not satisfied. This is because for each $L \in \mathcal{L}$, $\sup L = a \in L$. This counterexample shows that (C1) does not always imply (C2). We also see that $\mathcal{L}$ is not T_0-separating, because there does not exist $L \in \mathcal{L}$ that T_0-separates, say, $\frac{1}{4}$ and $\frac{1}{8}$. This shows that condition (C1) does not always imply T_0-separation.

Example 4.3. Let $X = (0, 1)$ and consider the nest $\mathcal{L} = \{(0, a) : a \in \mathbb{R}, \frac{1}{2} \leq a < 1\}$, on X. We remark that condition (C2) is satisfied, but condition (C3) is not satisfied. This is because for each $L \in \mathcal{L}$, $\sup L = a \notin L$; this shows that (C2) is satisfied. But we also see that there does not exist

$L \in \mathcal{L}$, such that $\sup L = \frac{1}{4} \in X - L$. This counterexample shows that (C2) does not always imply (C3) and also (C2) does not always imply that $\mathcal{L}$ is T_0-separating. Indeed, there does not exist $L \in \mathcal{L}$ that T_0-separates $\frac{1}{4}$ and $\frac{1}{8}$.

Remark 4.4. The results in both Examples 4.3 and 4.2 permit us to make some conclusions on the connection between T_0-separating nests and linear orders. It follows from the definition of nest and T_0-separation that a nest is T_0-separating, if and only if $\trianglelefteq_{\mathcal{L}}$ is a linear order. In addition, in Lemma 9 from [3], we get that if $<$ is a linear order on a set X, and $\mathcal{L}_< = \{(-\infty, a) : a \in X\}$, then $\mathcal{L}_<$ is T_0-separating. Why isn't the nest $\mathcal{L}$, in both of the above examples 4.3 and 4.2, T_0-separating? The answer lies on the fact that in the mentioned lemma from [3], the elements of the nest $\mathcal{L}_<$ satisfy an 1-1 correspondence with the elements of the set X, something that does not happen in our examples. So, the set X, in Examples 4.3 and 4.2 is not linearly ordered via $\trianglelefteq_{\mathcal{L}}$.

Example 4.5. Let $X = \{a, b\}$ and consider the nest $\mathcal{L} = \{\{a\}\}$, on X. We remark that $\mathcal{L}$ is T_0-separating. Indeed, since $a \neq b$, there exists $L = \{a\} \in \mathcal{L}$, such that $a \in \{a\}$ and $b \notin \{a\}$. We remark that (C3) is not satisfied though. Indeed, for $b \in X$, there does not exist $L \in \mathcal{L}$, such that $\sup L = b$. We observe that $L = \{a\} \in \mathcal{L}$ and that $\sup L = a$.

Example 4.6. Consider $X = \mathbb{R}$ and the nest $\mathcal{L} = \{(-\infty, a] : a \in \mathcal{R}\}$, on $\mathbb{R}$. One can easily see that $\mathcal{L}$ T_0-separates $\mathbb{R}$. But, for each $L \in \mathcal{L}$, we have that $\sup(-\infty, a] = a \in L$. So, property (C2) is not satisfied. With this example we see that the T_0-separation property of $\mathcal{L}$ does not necessarily imply property $(C2)$.

Example 4.7. Let $X = \mathbb{R}$ and let $\mathcal{L} = \{(a, \infty) : a \in \mathbb{R}\}$. One can easily deduce that $\mathcal{L}$ T_0-separates $\mathbb{R}$, but property (C1) is not satisfied, since for each $L \in \mathcal{L}$ there does not exist $\sup L$, with respect to $\trianglelefteq_{\mathcal{L}}$. So, the T_0-separation property of $\mathcal{L}$ does not necessarily imply property (C1).

We will now prove that property (C3) implies the T_0-separation of $\mathcal{L}$.

Proposition 4.8. *Let X be a set and let $\mathcal{L}$ be a nest on X that satisfies property (C3). Then, $\mathcal{L}$ T_0-separates X.*

Proof. Let $x \neq y \in X$. By (C3), there exists $L_x \in \mathcal{L}$, such that $\sup L_x = x$ and there also exists $L_y \in \mathcal{L}$, such that $\sup L_y = y$. Since $\mathcal{L}$ is a nest on X,

we have that either $L_x \subset L_y$ or $L_y \subset L_x$. If $L_x \subset L_y$, then $\sup L_x \trianglelefteq_{\mathcal{L}} \sup L_y$, which implies that $x \triangleleft_{\mathcal{L}} y$. If $L_y \subset L_x$, we have that $\sup L_y \trianglelefteq_{\mathcal{L}} \sup L_x$, which implies that $y \triangleleft_{\mathcal{L}} x$. So, either $x \triangleleft_{\mathcal{L}} y$ or $y \triangleleft_{\mathcal{L}} x$, proving that $\mathcal{L}$ T_0-separates X. $\square$

Lemma 4.9. *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$ be a nest.*

1. *If condition (C1) is satisfied and* $\sup L = k$*, then $L \supset X - \uparrow k$.*
2. *If condition (C2) is satisfied and* $\sup L = k$*, then $L \subset X - \uparrow k$.*

Proof. 1. Let $L \in \mathcal{L}$ and let $k = \sup L \in X$. Then, for each $x \in L$, $x \trianglelefteq_{\mathcal{L}} k$. Let $y \in X - L$. Since $x \in L$ and $y \notin L$, we have that $x \triangleleft_{\mathcal{L}} y$, for each x. So, $k \trianglelefteq_{\mathcal{L}} y$, and so $y \in \uparrow k$. Thus, for each $y \in X - L$, we have that $y \in \uparrow k$. The latter gives that $X - L \subset \uparrow k$, which implies that $L \supset X - \uparrow k$.

2. For each $x \in L$, we have $x \triangleleft_{\mathcal{L}} k$, so $k \ntrianglelefteq x$ [3], which implies that $x \in X - \uparrow k$. Thus, $L \subset X - \uparrow k$. $\square$

From now on, $\mathcal{T}_{\mathcal{L}}$ will denote the topology generated by the nest $\mathcal{L}$, on X, and $\mathcal{T}_l$ the lower topology on X.

Proposition 4.10. *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$ be a nest. If condition (C2) is satisfied, then:*

1. $L = X - \uparrow k$*, where* $k = \sup L$*, with respect to* $\triangleleft_{\mathcal{L}}$*, for each* $L \in \mathcal{L}$.
2. $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$.

Proof. 1. follows by Lemma 4.9.

2. As we have seen in section 3, a subbase for $\mathcal{T}_l$ is of the form $\mathcal{S} = \{X - \uparrow k : k \in X\}$. Let $L \in \mathcal{L}$. Part 1. gives that $L = X - \uparrow k$, so $L \in \mathcal{T}_l$ and the result follows. $\square$

Theorem 4.11. *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$ be a nest on X, such that condition (C3) is satisfied. Then, $\mathcal{T}_{\mathcal{L}} = \mathcal{T}_l$.*

Proof. Proposition 4.10 gives that $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$. We now consider a subbasic open set of $\mathcal{T}_l$ of the form $X - \uparrow x$. Then, there exists $L \in \mathcal{L}$, such that $\sup L = x$. But, according to Proposition 4.10, $L = X - \uparrow x$. So, $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$ and the statement of the theorem follows. $\square$

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[3] Indeed, if $k = x$ we get a contradiction. If $x \triangleleft_{\mathcal{L}} k$, then there exists $L_1 \in \mathcal{L}$, such that $x \in L_1$ and $k \notin L_1$. If $k \triangleleft_{\mathcal{L}} x$, then there exists $L_2 \in \mathcal{L}$, such that $k \in L_2$ and $x \notin L_2$. But $\mathcal{L}$ is a nest. If $L_1 \subset L_2$, then $x \notin L_1$ and $x \in L_1$, a contradiction. If $L_2 \subset L_1$ we get a contradiction in a similar way.

Remark 4.12. Let $\mathcal{L}$ be a nest on a set X. Let $\mathcal{R}$ be another nest on X, such that there exists a mapping from $\mathcal{L}$ to $\mathcal{R}$, so that $x \triangleleft_{\mathcal{L}} y$, if and only if $y \triangleleft_{\mathcal{R}} x$. So, $x \triangleleft_{\mathcal{L}} y$, if and only if there exists $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$, if and only if there exists $R \in \mathcal{R}$, such that $y \in R$ and $x \notin R$.

Note that Theorem 2.6, from [3] requests $\mathcal{L} \cup \mathcal{R}$ to form a T_1-separating subbase for X; here we do not demand this, so neither $\mathcal{L}$ nor $\mathcal{R}$ will necessarily T_0-separate X. We keep only the dual order-theoretic properties of these two nests, but we do not necessarily keep the property that restricts them on a line. So, we are now able to rewrite for $\mathcal{R}$, in a dual way, the properties that hold for $\mathcal{L}$.

Definition 4.13. Let X be a set and let $\mathcal{L}$ and $\mathcal{R}$ be two nests on X, that satisfy the properties of Remark 4.12. We call such nests *dual nests*. $\mathcal{L}$ will be called *dual to* $\mathcal{R}$ and $\mathcal{R}$ dual to $\mathcal{L}$.

Let X be a set and let $\mathcal{R}$ be dual to the nest $\mathcal{L}$, where $\mathcal{L}$ satisfies properties (C1),(C2),(C3). In a similar fashion, we define the following properties for $\mathcal{R}$:

(C1)* For each $R \in \mathcal{R}$, there exists $\sup R$ with respect to $\trianglerighteq_{\mathcal{R}}$.
(Equivalently, for each $R \in \mathcal{R}$, there exists $\inf R$ with respect to $\trianglelefteq_{\mathcal{L}}$.)

(C2)* For each $R \in \mathcal{R}$, there exists $\sup R$ with respect to $\trianglerighteq_{\mathcal{R}}$, such that $\sup R \in X - R$.
(Equivalently, for each $R \in \mathcal{R}$ there exists $\inf R$ with respect to $\trianglelefteq_{\mathcal{L}}$, such that $\inf R \in X - R$).

(C3)* For each $x \in X$, there exists $R \in \mathcal{R}$, such that there exists $\sup R \in X - R$ with respect to $\trianglerighteq_{\mathcal{R}}$ and also property (C2)* holds.
(Equivalently, for each $x \in X$, there exists $R \in \mathcal{R}$, such that there exists $\inf R \in X - R$, with respect to $\trianglelefteq_{\mathcal{L}}$ and also property (C2)* holds).

One easily observes that Proposition 4.1 holds, too, if we substitute (C1)*, (C2)*, (C3)* in the place of (C1), (C2),(C3), respectively.

Proposition 4.10 can be also stated with respect to $\mathcal{R}$ in a dual way.

Proposition 4.14. *Let X be a set and let $\mathcal{R} \subset \mathcal{P}(X)$ be a nest. If condition (C2)* is satisfied, then:*

1. *$R = X - \uparrow k$, where $k = \sup R$ with respect to $\trianglerighteq_{\mathcal{R}}$ for each $R \in \mathcal{R}$ (or, equivalently, $R = X - \downarrow k$, where $k = \inf R$ with respect to $\trianglelefteq_{\mathcal{L}}$).*

2. $\mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_U$.

In a similar way, we can restate Theorem 4.11, with respect to $\mathcal{R}$.

Theorem 4.15. *Let X be a set and let $\mathcal{R} \subset \mathcal{P}(X)$ be a nest on X, such that condition (C3)* is satisfied. Then $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_U$.*

We can now sum up Theorems 4.11 and 4.15, in the following theorem.

Theorem 4.16. *Let X be a set and let $\mathcal{L}$ and $\mathcal{R}$ be two dual nests on X.*

1. *If $\mathcal{L}$ satisfies (C2) and if $\mathcal{R}$ satisfies (C2)*, then $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}} \subset \mathcal{T}_{in}^{\mathcal{L}}$.*
2. *If $\mathcal{L}$ satisfies (C3) and if $\mathcal{R}$ satisfies (C3)*, then $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}} = \mathcal{T}_{in}^{\mathcal{L}}$.*

As we can see in the two examples that follow, the conditions of statements 1. and 2. from Theorem 4.16 are necessary but not sufficient.

Example 4.17. Let $X = \{x_1, x_2\}$ and let $\mathcal{L} = \{\{x_1\}\}$. Then, $\mathcal{T}_{\mathcal{L}} = \{\{x_1\}, \{x_1, x_2\}, \emptyset\}$ is the topology on X which is generated by $\mathcal{L}$. We observe that $x_1 \lhd_{\mathcal{L}} x_2$. Then, $\uparrow x_1 = \{x_1, x_2\}$, $X - \uparrow x_1 = \emptyset$, $\uparrow x_2 = \{x_2\}$ and $X - \uparrow x_2 = \{x_1\}$. So, the lower topology $\mathcal{T}_l = \{\emptyset, \{x_1\}, \{x_1, x_2\}\} = \mathcal{T}_{\mathcal{L}}$. Now, we define $\mathcal{R} = \{\{x_2\}\}$ and $x_2 \rhd_{\mathcal{R}} x_1$, if and only if there exists $R \in \mathcal{R}$, such that $x_2 \in R$ and $x_1 \notin R$. So, $x_1 \lhd_{\mathcal{L}} x_2$ if and only if $x_2 \rhd_{\mathcal{R}} x_1$. Then, $\mathcal{T}_{\mathcal{R}} = \{\{x_2\}, \{x_1, x_2\}\}$ is the topology on X which is induced by $\mathcal{R}$. Also, $\downarrow x_1 = \{x_1\}$, $\downarrow x_2 = \{x_1, x_2\}$, $X - \downarrow x_1 = \{x_2\}$ and $X - \downarrow x_2 = \emptyset$. So, the upper topology $\mathcal{T}_U = \{\emptyset, \{x_2\}, \{x_1, x_2\}\} = \mathcal{T}_{\mathcal{R}}$.

From the above, we conclude that $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}} = \mathcal{T}_{in}^{\mathcal{L}}$ is equal to the discrete topology, although property (C3) is not satisfied. This is because for $x_1 \in X$, there is no $L \in \mathcal{L}$ such that $x_1 = \sup L$; here $\sup L = x_2$.

Example 4.18. Let $X = \{x_1, x_2, x_3, x_4\}$ and let $\mathcal{L} = \{\{x_1, x_2\}, \{x_1, x_2, x_3, x_4\}\}$. Then, one can easily see that $x_2 \lhd_{\mathcal{L}} x_3$, $x_2 \lhd_{\mathcal{L}} x_4$, $x_1 \lhd_{\mathcal{L}} x_3$ and $x_1 \lhd_{\mathcal{L}} x_4$. Also, $\uparrow x_1 = \{y \in X : x_1 \unlhd_{\mathcal{L}} y\} = \{x_1, x_3, x_4\}$ and $X - \uparrow x_1 = \{x_2\}$. Similarly, $\uparrow x_2 = \{x_2, x_3, x_4\}$ and $X - \uparrow x_2 = \{x_1\}$; $\uparrow x_3 = \{x_3\}$ and $X - \uparrow x_3 = \{x_1, x_2, x_4\}$; $\uparrow x_4 = \{x_4\}$ and $X - \uparrow x_4 = \{x_1, x_2, x_3\}$. The lower topology now takes the form $\mathcal{T}_l = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_3, x_4\}\}$ and $\mathcal{T}_{\mathcal{L}} = \{\emptyset, \{x_1, x_2\}, \{x_1, x_2, x_3, x_4\}\}$. So, $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$, but $\mathcal{L}$ is not T_0-separating, because $x_3 \neq x_4$ and there is no $L \in \mathcal{L}$ that T_0-separates x_3 and x_4. Also, $\mathcal{L}$ does not satisfy property (C2), because $\sup\{x_1, x_2\}$ does not exist.

Now, we consider $\mathcal{R} = \{\{x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}$, and we observe that $x_3 \rhd_{\mathcal{R}} x_1, x_3 \rhd_{\mathcal{R}} x_2, x_4 \rhd_{\mathcal{R}} x_2$ and $x_4 \rhd_{\mathcal{R}} x_3$. So, there exists a mapping between the nests $\mathcal{L}$ and $\mathcal{R}$, and their duality can be seen from the fact that $x_3 \rhd_{\mathcal{R}} x_1$ iff $x_1 \lhd_{\mathcal{L}} x_3$, $x_3 \rhd_{\mathcal{R}} x_2$ iff $x_2 \lhd_{\mathcal{L}} x_3$, $x_4 \rhd_{\mathcal{R}} x_2$ iff $x_2 \lhd_{\mathcal{L}} x_4$ and $x_4 \rhd_{\mathcal{R}} x_1$ iff $x_1 \lhd_{\mathcal{L}} x_4$. It can be easily deduced that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}$ and that the upper topology is $\mathcal{T}_U = \{\emptyset, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_3\}, \{x_4\}, \{x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}$. Also, $\mathcal{R}$ is not T_0-separating, neither satisfies property (C2)* and we deduce that $\mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_U$. Last, but not least, we see that $\mathcal{T}_{in}^{\mathcal{L}}$ is the discrete topology, thus $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}} \subset \mathcal{T}_{in}^{\mathcal{L}}$.

5. Conclusions and Questions.

In Remark 3.1 we stated that a non-reflexive order that is induced by a nest $\mathcal{L}$ makes $\mathcal{T}_{in}^{\mathcal{L}}$ equal to the discrete topology, so it will automatically be finer than $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$. If the order is reflexive, then Theorem 4.16 shows that there is a case where $\mathcal{T}_{in}^{\mathcal{L}}$ is equal to $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$, and this is when properties (C3) and (C3)* are both satisfied. But (C3) (resp. (C3)*) implies that $\mathcal{L}$ (resp. $\mathcal{R}$) is T_0-separating, while in Example 4.5 (and Proposition 4.1) we see that $\mathcal{L}$ can be T_0-separating, without (C3) being satisfied. So, the two topologies coincide in certain type of spaces that are T_0-separating under properties (C3) and (C3)*.

The real line, with its natural topology that is generated by the nests $\mathcal{L} = \{(-\infty, a) : a \in \mathbb{R}\}$ and $\mathcal{R} = \{(a, \infty) : a \in \mathbb{R}\}$ is a specific example of a space of the type that is described in Theorem 4.16 2. **Question**: are there other LOTS, apart from the real line with its natural order, such that 2. from Theorem 4.16 is satisfied?

Furthermore, we remark that if property (C2) alone is satisfied, then for each $L \in \mathcal{L}$ we have that $\sup L \in X - L$, so that $\sup L \notin L$. So, for each $L \in \mathcal{L}$ there is no $\unlhd_{\mathcal{L}}$-maximal element in L, because for each $L \in \mathcal{L}$ there exists $k \in L$, $x \unlhd_{\mathcal{L}} k$, for each $x \in L$, so that $k = \sup L$. In a similar fashion, we can obtain a dual property for the dual nest $\mathcal{R}$, with the ordering $\unrhd_{\mathcal{R}}$. We will use this remark in order to find conditions which imply the orderability problem that was introduced by J. van Dalen and E. Wattel, in [2].

Theorem 5.1. *Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests on X, such that $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}}$. Let also properties (C3) and (C3)* be satisfied. Then, X is a LOTS.*

Proof. In section 1. we stated the characterization of LOTS that was introduced by van Dalen and Wattel. We observe that property (C3) (similarly

(C3)*) implies T_0-separation and interlocking (see Theorem 2.8), so that the conditions of van Dalen and Wattel follow immediately and so X is a LOTS. $\square$

Property (C3) (resp. (C3)*) implies naturally T_0-separation and interlocking. Property (C2) (resp. (C2)*) implies interlocking, if we add T_0-separation. So, we can restate Theorem 5.1 as follows:

Theorem 5.2. *Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests on X, such that $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}}$ and each of $\mathcal{L}$ and $\mathcal{R}$ T_0-separates X, respectively. Let also properties (C2) and (C2)* be satisfied. Then, X is a LOTS.*

Question: What is the difference between LOTS that are implied by Theorem 5.1 from LOTS being implied by Theorem 5.2? Are there distinct examples of such spaces, spotting the difference between these properties?

The following example shows that both properties (C2) and (C2)* do not necessarily imply T_0-separation. So, Theorem 5.2 without the T_0-separation property of $\mathcal{L}$ and $\mathcal{R}$ generates spaces that are not necessarily linearly ordered, but carry analogous order theoretic properties to linearly ordered sets.

Example 5.3. Consider the set of real numbers $\mathbb{R}$ and the nests $\mathcal{L} = \{(-\infty, n) : n \in \mathbb{N}\}$ and $\mathcal{R} = \{(n, \infty) : n \in \mathbb{N}\}$ on $\mathbb{R}$. Then, $\mathcal{L}$ and $\mathcal{R}$ satisfy conditions (C2) and (C2)*, respectively. Indeed, for each $L = (-\infty, n) \in \mathcal{L}$, $\sup L = n \notin L$ and for each $R \in \mathcal{R}$, $\inf(n, \infty) = n \notin R$. We also remark, from the definition of T_0-separation, that neither $\mathcal{L}$ nor $\mathcal{R}$ are T_0-separating.

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